ONLINE APPENDIX

List of mathematical symbols and notation used in text.--

$\in$ - a member of a set, for example, $x \in G$ reads as “x is a member or element in the set G”.

$|X|$ - the size or cardinality of a set X, that is, the number of members of X.

$\binom{x}{y}$ - the binomial coefficient, expanded as $\frac{x!}{y!(x-y)!}$

a_i^x - a parameter of a species x in guild i.

$\forall$ - for all.

Ensemble size.-- The number of *SLN*s that can be derived from a metanetwork ensemble is finite, because there is a finite number of arrangements or graphs of the species. The number or ensemble size also defines the maximum variation possible for a real community, unless the taxon composition itself changes. The number of SLNs (the number of elements in S, or the cardinality of S) is designated $|S|$. Examine the simple metanetwork in Figure 10. We will designate the metanetwork U, and the number of guilds as $|U|$. Guilds 1 and 2 (G1 and G2) comprise species that are preyed upon by species in G4. In order to construct a SLN, we must specify exactly which species in G1 and G2 are preyed upon by each species in G4. Let the presence of a metanetwork link be indicated by elements a_{ij} of the adjacency matrix, being one if a link exists, and zero otherwise. Then the maximum number of prey species (maximum in-degree) available to any species in G4, denoted r_{max} is

$$r_{max}(G_4)=a_{31}|G_1|+a_{32}|G_2|+a_{33}|G_3|+a_{34}|G_4|=a_{31}|G_1|+a_{32}|G_2|=5$$

since a_{33} and a_{34} equal zero. This can be generalized to a metanetwork of any complexity as

$$r_{max}(G_i)=\sum_{j=1}^{|U|} a_{ij}|G_j|$$

Since every species in G4 may have an in-degree range of one to r_{max}, and every one of these possibilities or predatory states could be combined with every other state of the remaining species in G4, the maximum number of possible networks is simply $r_{max}^{|G4|}$. Moreover, the in-degree or number of predatory states of every species in the network may be combined regardless of guild membership, allowing us to generalize to metanetworks of all sizes and complexity. By this argument, the number of possible SLNs, |S|, is the product of the number of predatory states of species in every guild, that is,

$$|S| = \prod_{i=1}^{|U|} \sum_{j=1}^{|U|} \left(a_{ij} |G_j| \right)^{|G_i|}$$

This formula overestimates |S| because prey species in a guild are treated neutrally from a consumer's point of view. Neutral is used here in the sense of ecological neutrality; the species are indistinguishable from each other on the basis of trophic properties. In other words, the above calculation of |S| does not specify *which* prey species are linked to when the predatory states of different predators are combined. Therefore, many of the combinations counted in the calculation will be isomorphic food webs, and they would not have unique ecological properties.

In order to resolve this problem, and gain a more accurate measure of |S|, we need to understand the number of different ways in which a consumer's links can be distributed among it's prey. This is a classic partitioning problem, where say we wish to determine the number of ways in which n fossils can be distributed among m museum drawers, with k_1 fossils in the first drawer, k_2 in the second, and so on. The fossils (links) are not distinct, and we do not care specifically to which drawer (prey species and guild) they are assigned. The trick is to first state the problem as: How many combinations of n fossils can I get if I have m drawers to select from? Or, how many combinations of r links can I get if there are g guilds to select from? We recognize that we are in fact permuting n fossils plus m-1 partitions among the drawers, yielding

$$\binom{n+m-1}{m}$$

Therefore in our sample food web (Fig. 10), if a species in G4 has three in-links, then the problem is

$$\binom{3+2-1}{3}=\binom{4}{3}=4$$

The links can be partitioned between guilds G1 and G2 as {3,0}, {2,1}, {1,2} or {0,3}. None of these are isomorphic topologies.

The calculation of |S| can now be refined, where the topologies obtained for a particular in-degree are combined with those of other species, rather than simply combining the number of in-degrees possible. We proceed in several steps. First, determine the maximum number of in-links possible for a species in guild G_i. Next, determine the number of in-link topologies possible for each in-degree (1 to r_{max}), given the set of prey guilds,

$$t_i^x=\sum_{r_x=1}^{r_{max}}\binom{r_i^x+a_i-1}{a_i}$$

where t_i^x is the number of topologies possible for species x_i, which is a member of guild G_i ($x_i \in G_i$), r_i^x is the in-degree of x_i, and a_i is the number of guilds upon which G_i preys (sum of the G_i^{th} row of the adjacency matrix). Note that this is calculated and summed over all possible in-degree values, one to r_{max}. Following from the earlier calculation of |S|, the number of topological combinations among species in G_i is $\left(t_i^x\right)^{|G_i|}$. We therefore re-calculate |S| as

$$|S|=\prod_{i=1}^{|U|}\left(t_i^x\right)^{|G_i|}$$

This calculation is still an overestimate, however, because the number of links between a consumer species and any prey guild is unconstrained. It represents all elements in the power set of S. In terms of the museum fossils analogy, we have assumed that the cabinet drawers are of infinite capacity (sadly, a curatorial fantasy). A more accurate measure of |S| is possible if we limit drawer capacity to some finite number of fossils, thereby limiting ourselves to the set *F* of real possibilities. The food web situation is more complicated because different prey guilds will most likely have different species-richnesses, and hence differing capacities for links. The situations would be analogous if drawers in the collection were of different sizes, indeed a curatorial nightmare! The solution would be to modify the above formula for t_i^x, using only topologies where the number of links from a prey guild to x_i is less than or equal to the species-richness of the prey guild. That is, the capacity of the prey guild is not exhausted. This solution, however, requires partitioning r_x appropriately among the prey guilds so that this condition is met. The set of all such partitions that match the constraints of prey guild species-richnesses can be determined, but the solution is not straightforward and requires application of a partition function and partition theory. Both those topics are, unfortunately, beyond the scope of the current paper. Therefore, the given calculation of |S| remains an overestimate at this point.

FURTHER READING

Comprehensive but accessible treatments of counting, combinatorics and probability may be found in Goldberg (1960) and Riordan (1958). The topics of networks, network theory and complex systems are currently very popular and there are a large number of popular treatments. Several which I recommend are Barabási (2003), Watts (2003) and Mitchell (2009).

LITERATURE CITED

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